$$\lambda(a_0a_2-a_1^2)+\mu(a_0a_3-a_1a_2)+\nu(a_1a_3-a_2^2)=0,$$

while the common intersection of all these surfaces, the given twisted cubic, is converted *point by point* into itself.

To find other special configurations under this transformation, we examine the curves of intersection (other than the twisted cubic common to them all) of the cubic surfaces $b_0=0$, $b_1=0$, $b_2=0$, $b_3=0$ taken two at a time. We find:

$b_0=0$ and $b_1=0$ are tangent to each other along the line ($a_0=0$, $a_1=0$), which is moreover a double line of the surface $b_0=0$;

$b_2=0$ and $b_3=0$ are, similarly, tangent to each other along the line ($a_2=0$, $a_3=0$), which is a double line of the surface $b_3=0$;

$b_0=0$ and $b_2=0$ have in common the line ($a_0=0$, $a_1=0$), which of course is a double intersection, and also the line ($a_1=0$, $a_3=0$);

$b_1=0$ and $b_3=0$, similarly, have in common as double intersection the line ($a_2=0$, $a_3=0$) and as simple intersection the line ($a_0=0$, $a_2=0$);

$b_1=0$ and $b_2=0$ have in common the three lines ($a_0=0$, $a_1=0$), ($a_1=0$, $a_2=0$) and ($a_2=0$, $a_3=0$); and, finally,

$b_0=0$ and $b_3=0$ have in common a second twisted cubic, given parametrically by the relation

$$(6)\ldots.x_0 : x_1 : x_2 : x_3=2\lambda^3 : \lambda^2 : -\lambda : -2$$

It follows immediately (and the results can be verified without difficulty) that: the line ($a_0=0$, $a_1=0$) is converted into the point (0 : 0 : 0 : 1), and the line ($a_2=0$, $a_3=0$) into the point (1 : 0 : 0 : 0);

The lines ($a_1=0$, $a_3=0$) and ($a_0=0$, $a_2=0$) are interchanged; the line ($a_1=0$, $a_2=0$) is converted into itself by an involution in which corresponding pairs of points are harmonically conjugate with respect to the double points (1 : 0 : 0 : 0) and (0 : 0 : 0 : 1);

While the line ($a_0=0$, $a_3=0$) is converted into the twisted cubic (6), and vice versa.

Finally, it is geometrically evident that every point of any tangent to the twisted cubic is transformed into the point of contact of that tangent, and hence that the quartic developable (5) is transformed into the original twisted cubic, its cuspidal edge.

BERKELEY, CALIFORNIA, *December 24, 1901.*

DETERMINATION OF THE GROUP OF RATIONALITY OF A LINEAR DIFFERENTIAL EQUATION.

By DR. SAUL EPSTEEN.

1. When a special algebraic equation is given, its group G can be theoretically determined: 1° by constructing an $n!$-valued function and finding the Galois resolvant; or, 2°, by applying the characteristic double-property of the group, that every rational function of the roots which remains unaltered by all the substitutions of G lies in the domain of rationality; and conversely. Both of these methods are generally impracticable, and in practice some device is resorted to in order to obtain the group. When a single rational function is known the following theorem has been found very useful:* "If a rational function $\phi(x_1 \ldots . x_n)$ remains formally unaltered by the substitution of a group G' and by no other substitutions, and if ϕ equals a quantity lying in the domain of rationality R, and if the conjugates of ϕ under $G_{n!}$ are all distinct, then the group of the given equation for the domain R is a subgroup of G'."

A similar situation confronts us when we deal with linear differential equations. When a special linear differential equation is given, its group may be found: 1° by constructing the Picard resolvant†, or, 2° by applying the Picard-Vessiot characteristic double-property of the group‡, that every rational differential function (of a fundamental system) of the integral which remains unaltered, as a function of x, by all the transformations of G lies in R (the domain of rationality); and conversely. As in the case of algebraic equations both of these methods are impracticable, and in order to find the group a device must be resorted to. With this end in view, I will prove the analog of the above theorem of Algebra, viz: *If a rational differential function (of a fundamental system) of integrals $\psi(y_1 \ldots . y_n, y'_1 \ldots . y'_n, \ldots .)$ remains formally unaltered by the transformations of a complex r-parameter linear homogeneous group G_r; and if, when ψ is transformed by the most general linear homogeneous transformation, it depends on $n^2 - r$ essential parameters§, then the group G of the given equation is a subgroup of G_r.*

The advantage of this theorem is that only one function ψ is required, whereas the Picard-Vessiot double theorem relates to the entire infinity of rational functions.

2. Let the given differential equation be

$$(1) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots . + p_n y = 0$$

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* L. E. Dickson, *Theory of Algebraic Equations* (John Wiley and Sons).

† Schlesinger's *Handbuch der Theorie der linearen Differentialgleichungen*, II, part 1, pp. 60 and 68.

‡ Schlesinger, *Handbuch*, II, 1, p. 71.

§That is to say, if $G_{n^2} : \bar{y}_i = \sum_{k=1}^{n} a_{ik} y_k$ is the most general transformation, then $\psi(\bar{y}) = \psi(\Sigma a_{ik} y_k) = \phi(y, A_1 \ldots . A_{n^2-r})$ where the $n^2 - r$ A's are all essential. Cf. Picard, *Traité d' Analyse*, III, p. 508; also p. 519.

which is supposed to be special, *i. e.* the coefficients p are known (and not indeterminate) functions of x. A domain of rationality R is supposed to be assigned, this domain containing at least the coefficients. The group of rationality of the differential equation (1) in (R) is in general a complex group made up of ν systems of linear homogeneous transformations

$$G_r : \bar{y}_{ij} = \sum_{k=1}^{n} a_{ikj} y_{kj} \qquad \begin{pmatrix} i=1....n \\ j=0,\ 1....\nu-1 \end{pmatrix}.$$

$\phi(y)=a(x)$ is *formally* invariant under this r-parameter group. This group G_r contains a certain maximal continuous subgroup Γ which is generated by infinitesimal transformations

$$\Gamma : \bar{y}_{io} = \sum_{k} a_{iko} y_{ko} \qquad (i=1....n).$$

If $T_0=I,\ T_1,\ ....T_{\nu-1}$ are the transformations for which

$$T_j^{-1}\,\Gamma\,T_j=\Gamma \qquad (j=0,\ 1....\nu-1),$$

then G_r is of the form* $G_r : \Gamma,\ T_1\Gamma,\ ....,\ T_{\nu-1}\Gamma$.

Moreover, $T_o=I,\ T_1....,\ T_{\nu-1}$ form a substitution-group.

Suppose now that $\phi(y)=r(x)$ is any differential function (as usual, of a fundamental system) of integrals of equation (1), which remains *numerically* unaltered (*i. e.* as function of x) under G_r. It follows that $\phi(y)$ is *formally* unaltered under Γ.† Under the group G_r, ϕ will therefore take ν values: $\phi=\phi_0,\ \phi_1$ $....\phi_{\nu-1}$, and the function

$$\phi=\frac{1}{\nu}(\phi_0+\phi_1+....+\phi_{\nu-1})$$

remains formally invariant under G_r. It follows now by the Lagrange-Vessiot theorem‡ that ϕ is a rational function of ψ and is therefore a quantity lying in R (*i. e.* $r(x)$ lies in R).

3. We know now that our group G_r has the property that every rational differential function of a fundamental system of integrals $y_1,\ ....,\ y_n$, which remains numerically unaltered by the most general transformation of G_r lies in the domain of rationality R. We will next show that G_r contains as a subgroup (or coincides with) the group of rationality G of the equation (1). Writing

$$(2) \qquad u(y)=A_1y_1+....+A_ny_n$$

* Schlesinger, *Handbuch*, II, 1, p. 79.

† Schlesinger, *Handbuch*, II, 1, p. 79, calls θ a characteristic invariant of Γ; cf. also Gino Fano, *Mathematische Annalen*, Vol. 53, p. 403.

‡ Vessiot: *Annales de l' Ecole Normale Superieure*, 1892, p. 223; Schlesinger, II, 1, pp. 53, 57; Picard, *Traite d' Analyse*, III, pp. 521-2.

(the A's being undetermined functions of x) we differentiate n^2 times and eliminate the derivatives of the nth and higher orders by means of the differential equation (1). This gives us n^2+1 equations between the n^2 quantities $y_1 \ldots y_n$, $y'_1 \ldots y'_n$, $y_1^{(n-1)} \ldots y_n^{(n-1)}$; eliminating we obtain a linear differential equation

$$(3) \qquad \frac{d^{n^2}u}{dx^{n^2}} + \pi_1 \frac{d^{n^2-1}u}{dx^{n^2-1}} + \ldots + \pi_{n^2} u = 0.$$

We assume that

$$(4) \qquad \theta(u) = 0$$

is a non-linear differential equation of the lowest order with the following properties: its coefficients are rational functions of the A's, their derivatives, and of x; it is irreducible in the sense of Koenigsberger;* at least one of its integrals satisfies (3).†

The equation (2) was differentiated n^2 times; neglecting the results of the last differentiation, we have n^2 equations in the n^2 quantities $y_1 \ldots y_n$, $y'_2 \ldots y'_n$, $\ldots$, $y_1^{(n-1)} \ldots y_n^{(n-1)}$. Solving these equations, we obtain

$$y_j^{(i)} = w_{ji1} u + w_{ji2} \frac{du}{dx} + \ldots + w_{jin^2} \frac{d^{n^2-1}u}{dx^{n^2-1}}.$$

Substituting the values thus obtained in $\phi(y_1 \ldots y_n, \ldots, y_1^{(n-1)} \ldots y_n^{(n-1)}) = r(x)$ we obtain a rational differential expression

$$(5) \qquad F\left(u, \frac{du}{dx} \ldots \frac{d^\nu u}{dx^\nu}\right) = r(x) \qquad (\nu \leqq n^2 - 1).$$

The equation (4) having an integral in common with (5), all of the other integrals of (4) must also satisfy (5).‡ Thus, when we replace u in (5) by any integral of (4), F remains always equal to $r(x)$. The group of rationality of (1) is the totality of transformations which correspond to the passage from some particular integral u_1 of

$$(4) \qquad \theta(u) = 0$$

to all the other integrals of this equation.§ We saw however that $F = r(x)$ for all the integrals of $\theta(u) = 0$, hence $\phi(y)$ is numerically invariant under the group of rationality of the equation (1). If

$$\theta(u) \equiv F(u) - r(x) = 0$$

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* Schlesinger, *Handbuch*, II, 1, p. 66.

† *Handbuch*, II, 1, p. 65. In a letter to Picard (*Bulletin des Sciences Mathem.*, March, 1902) Professor Alfred Loewy proved the important theorem that the various equations (4) which are satisfied by the fundamental integrals of (3) are all of the *same* order.

‡ Since (4) is irreducible. Cf. Picard, *Traite*, III, p. 525.

§ *Handbuch*, II, 1, pp. 68-9.

G and G_r will coincide, but in general F may well be of higher order or higher degree than θ and consequently G will be a subgroup of G_r.

4. In the enunciation of the theorem I added that when $\psi(y)$ was transformed by the most general linear homogeneous transformation (in n variables and n^2 parameters) it would become a function of n^2-r essential parameters. This corresponds to the analogous requirement in Algebra that the conjugates of $\psi(x_1,\ldots.x_n)$ under the total symmetric group shall be distinct. The reason for this condition is that it enters in the proof of the Lagrange-Vessiot theorem which was employed in 2.

5. Example 1°: Consider the equation

$$(6) \qquad \frac{d^2y}{dx^2}+y=0.$$

A fundamental pair of integrals is

$$y_1=\sin x,\ y_2=\cos x;$$

between these exists the relation

$$(7) \qquad \psi(y)\equiv y_1^2+y_2^2=1.$$

Now $\psi(y)$ remains formally unaltered under the group G_1:

$$\begin{cases} \bar{y}_1=y_1\cos\alpha-y_2\sin\alpha, \\ \bar{y}_2=y_1\sin\alpha+y_2\cos\alpha; \end{cases} \qquad \begin{cases} \bar{y}_1=y_1\cos\alpha+y_2\sin\alpha, \\ \bar{y}_2=-y_1\sin\alpha+y_2\cos\alpha. \end{cases}$$

When ψ is transformed by

$$\bar{y}_1=a_{11}y_1+a_{12}y_2,\ \bar{y}_2=a_{21}y_1+a_{22}y_2,$$

there results $\psi(\bar{y})\equiv(a_{11}y_1+a_{12}y_2)^2+(a_{21}y_1+a_{22}y_2)^2$, a function of *three* (n^2-r) essential parameters, namely, $A_1y_1^2+A_2y_2^2+A_3y_1y_2$, where $A_1=a_{11}^2+a_{21}^2$, $A_2=a_{12}^2+a_{22}^2$, $A_3=2(a_{11}a_{12}+a_{21}a_{22})$. The group of rationality of (6) is therefore either G_1 or the identity. In the domain* of the trigonometric functions (or their equivalents, such as the exponential functions) the group of rationality of (1) is the identity. In a domain not including the trigonometric functions, G_1 is the required group.

Example 2°: The question of numerical invariance of the rational functions of the integrals was first raised by Klein.† Previous to this Picard‡ had given:

* For differential equations the domain of rationality is composed of the four fundamental operations, extraction of roots, differentiation and any functions which may be decided on arbitrarily. The functions are however so chosen as to always make the coefficients of the given equation rational.

† F. Klein, *Hohere Geometrie*, II, pp. 298-9. (B. G. Teubner, Leipzig).

‡ Picard, *Comptes Rendus*, 1883, p. 1134. Picard writes the second equation $Y_2=\sqrt{(1-\lambda^2)}y_1-\lambda y_2$; this cannot be correct since $\lambda=1$ gives $Y_2=-y_2$ instead of the identical transformation $Y_2=y_2$.

$$Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2$$
$$Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2$$

as the group of the equation

$$(8)\qquad x(1-x)\frac{d^2y}{dx^2}-\frac{x}{2}\frac{dy}{dx}+a^2y=0,\qquad (a=\text{constant})$$

since the relation between a suitably selected pair of fundamental integrals is $y_1^2+y_2^2=1$.*

On the basis of our theorem however, G_r becomes G_1:

$$\begin{cases} Y_1=\lambda y_1+\sqrt{(1-\lambda^2)}y_2, \\ Y_2=-\sqrt{(1-\lambda^2)}y_1+\lambda y_2, \end{cases} \qquad \begin{cases} \overline{Y}_1=\lambda y_1-\sqrt{(1-\lambda^2)}y_2, \\ \overline{Y}_2=\sqrt{(1-\lambda^2)}y_1+\lambda y_2; \end{cases}$$

and the group of rationality of (8) will be G_1 or the identity according to the domain of rationality which is selected. Since (8) is a special case of the hypogeometric equation

$$x(1-x)\frac{d^2y}{dx^2}+[Y-x(1+\alpha+\beta)]\frac{dy}{dx}-\alpha\beta y=0$$

the adjunction of the hypogeometric function would reduce the group of rationality to the identical transformation.

THE UNIVERSITY OF CHICAGO, *December, 1902.*

* Picard, *loc. cit.*

SOME FALLACIES IN TEXT-BOOKS ON ELEMENTARY SOLID GEOMETRY.

By PROF. G. W. GREENWOOD, McKendree College, Lebanon, Ill.

So much has been written on the fallacies encountered in elementary plane geometries in applying to curves theorems on the lengths of broken lines, that it is, perhaps, superfluous to note the more frequent errors of similar character in elementary texts on solid geometry.

For example, in the case of a cylindrical surface, there is in its definition no connection between it and a prismatic surface; and any attempt to *prove* that the surface of the inscribed or circumscribed prism approaches the surface of the cylinder as the number of sides is indefinitely increased, seems fallacious. For no matter how great is the number of sides of the inscribed prism, there is still an *infinite gap* between it and the cylinder in which it is inscribed, and to assume

that a relation between these surfaces, which initially is *entirely absent*, is supplied by multiplying the number of sides of the prism, is to deceive ourselves.

Equally vulnerable seems the corollary commonly attached, that the lateral area of the cylinder is therefore the limit of the lateral area of the inscribed prism when the number of sides is indefinitely increased; for such a corollary presupposes that we have already defined the *area* of a *curved surface* in some other way.

Such unjustifiable assumptions may be avoided by first showing that the area of the inscribed or circumscribed prism has a limit when the number of sides is indefinitely increased, and then *defining* the area of the cylinder as the limit of the area of the inscribed or circumscribed cylinder.

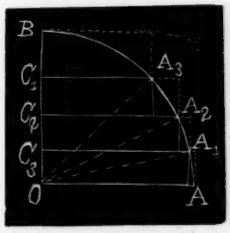
THE VOLUME OF THE SPHERE.

By HENRY L. COAR, A. M., University of Illinois.

The solids of revolution, that is, the solids generated by the revolution of a plane figure about an axis, offer many interesting problems to the student of synthetic geometry of three dimensions. The question of obtaining the volumes and total areas of solids, generated by the revolution of rectilinear figures, presents no particular difficulty, but when we attempt to find by purely synthetic means the volumes and superficial area of solids formed by curvilinear figures, the problem is not always simple without making some assumptions regarding limits. A well-known problem of this kind is to find the volume and area of an anchor-ring. A most interesting problem in this line is that of obtaining the volume of a sphere regarded as a solid of revolution. In the following proof, which I have not been able to find published anywhere, no assumptions of any kind regarding the existence of the limits are necessary.

We need consider only the hemisphere, which is generated by the revolution, through 360°, of a quadrant of a circle about a radius.

Let AOB be the quadrant of a circle and let us divide the radius OB into n equal parts. Then construct a set of inscribed and a set of circumscribed rectangles as indicated in the figure.



If now we rotate the complete figure through 360° about the radius OB, the quadrant will generate a hemisphere, while each of the inscribed as well as each of the circumscribed rectangles will generate a right circular cylinder. Let us designate the sum of the volumes of the cylinders generated by the inscribed rectangles by V_1, that of the cylinders generated by the circumscribed rectangles by V_2, and the volume of the hemisphere by V. We will first prove that both V_1 and V_2 approach V as their limit as n increases indefinitely, and

will then find an expression for either of these and obtain its limiting value. To prove the first we have always $V_1 < V < V_2$, hence

$$V_2 - V < V_2 - V_1 \text{ and } V - V_1 < V_2 - V_1.$$

But $V_2 - V_1$ is the volume of the lowest circumscribed cylinder, *i. e.*

$$V_2 - V_1 = \frac{r}{n}.\pi r^2 = \frac{\pi r^3}{n}$$

where r is the radius of the circle. Hence

$$V_2 - V < \frac{\pi r^3}{n} \text{ and } V - V_1 < \frac{\pi r^3}{n}.$$

It follows therefore that each of these differences can be made less than any assignable positive quantity, and hence

$$\lim_{n=\infty} V_1 = V \text{ and } \lim_{n=\infty} V_2 = V.$$

We thus prove that our hemisphere is actually the limit of the figures in question.

Let us now obtain an expression, say for V_2, and find its limiting value. We see that V_2 is the sum of n right circular cylinders. Let us find their volumes and add. For convenience denote the volumes of these n cylinders, beginning with the lowest one, by $v_1, v_2, \ldots . v_n$. We shall then have

$$v_1 = \frac{r}{n}.AO^2.\pi = \frac{\pi r^3}{n},$$

$$v_2 = \frac{r}{n}.A_1C_1^2.\pi = \frac{\pi r}{n}\left(r^2 - \frac{r^2}{n^2}\right) = \frac{\pi r^3}{n}\left(1 - \frac{1}{n^2}\right),$$

$$v_3 = \frac{r}{n}.A_2C_2^2.\pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{2r}{n}\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{2^2}{n^2}\right),$$

. . . . . .

. . . . . .

$$v_n = \frac{r}{n}(A_{n-1}C_{n-1})^2.\pi = \frac{\pi r}{n}\left[r^2 - \left(\frac{n-1}{n}r\right)^2\right] = \frac{\pi r^3}{n}\left(1 - \frac{(n-1)^2}{n^2}\right).$$

Hence

$$V_2 = v_1 + v_2 + \ldots . + v_n = \frac{\pi r^3}{n}\left[1 + \left(1 - \frac{1}{n^2}\right) + \left(1 - \frac{2^2}{n^2}\right) + \left(1 - \frac{3^2}{n^2}\right) + \ldots .\right.$$

$$+\left(1-\frac{(n-1)^2}{n^2}\right)\Big]$$

$$=\frac{\pi r^3}{n}\Big[n-\Big(\frac{1}{n}\Big)^2-\Big(\frac{2}{n}\Big)^2-\Big(\frac{3}{n}\Big)^2-\ldots.-\Big(\frac{n-1}{n}\Big)^2\Big]$$

$$=\pi r^3\{1-\frac{1}{n^3}[1^2+2^2+3^2+\ldots.+(n-1)^3]\}.$$

Now the series $1^2+2^2+3^2+\ldots.+(n-1)^2$ is the sum of the squares of the first $n-1$ positive integers. The formula for the sum of the squares of the first n integers is given in any College Algebra under "piles of shot" and is

$$1^2+2^2+3^2+\ldots.+n^2=\frac{n(n+1)\,(2n+1)}{6}.$$

Hence

$$1^2+2^2+3^2+\ldots.+(n-1)^2=\frac{(n-1)n\,(2n-1)}{6}.$$

Substituting this in the expression for V_2 we have

$$V_2=\pi r^3\Big[1-\frac{1}{n^3}\cdot\frac{(n-1)n\,(2n-1)}{6}\Big]$$

$$V_2=\pi r^3\Big[1-\tfrac{1}{6}(1-\frac{1}{n}).1.(2-\frac{1}{n})\Big].$$

Now proceed to the limit and we have

$$V=\lim_{n=\infty} V_2=\pi r^3[1-\tfrac{1}{6}(1).1.(2)]=\pi r^3[1-\tfrac{1}{3}]=\tfrac{2}{3}\pi r^3$$

which is the well-known result.

DERIVATION OF FORMULA FOR Tan½A IN SPHERICAL TRIGONOMETRY.

By GEORGE R. DEAN, School of Mines, Rolla, Mo.

Applying Napier's Rules of Circular Parts to the triangle formed by the bisector of angle A, the radius of the inscribed circle and one of the sides, we see that $\sin(s-a)=\tan r\cot\frac{1}{2}A$. Hence, $\tan\frac{1}{2}A=\tan r/\sin(s-a)$, so that

$$\frac{\tan\frac{1}{2}A}{\tan\frac{1}{2}B}=\frac{\sin(s-b)}{\sin(s-a)}....(1).$$

The sine proportion gives

$$\frac{\mathrm{Tan}\frac{1}{2}(A+B)}{\mathrm{Tan}\frac{1}{2}(A-B)}=\frac{\mathrm{Tan}\frac{1}{2}(a+b)}{\mathrm{Tan}\frac{1}{2}(a-b)}....(2).$$

For convenience, put $\mathrm{Tan}\frac{1}{2}A=x$, $\mathrm{Tan}\frac{1}{2}B=y$, $\mathrm{Tan}\frac{1}{2}(a+b)=m$, $\mathrm{Tan}\frac{1}{2}(a-b)=n$, $\sin(s-b)=p$, $\sin(s-a)=q$. Then

$$\frac{\frac{x+y}{1-xy}}{\frac{x-y}{1+xy}}=\frac{m}{n}....(3),$$

$$\text{and}\ \frac{x}{y}=\frac{p}{q}....(4).$$

From (3), $\frac{1+xy}{1-xy}=\frac{x-y}{x+y}\cdot\frac{m}{n}$. From (4), $\frac{x-y}{x+y}=\frac{p-q}{p+q}$.

Therefore, $\frac{1+xy}{1-xy}=\frac{p-q}{p+q}\cdot\frac{m}{n}=\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}....(5).$

$$xy=\frac{\tan\frac{1}{2}(a+b)-\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)+\tan\frac{1}{2}c}=\frac{\sin(s-c)}{\sin s}....(6).$$

Multiplying (4) by (6),

$$x^2=\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}=\tan^2\tfrac{1}{2}A.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

CRITICISM ON SOLUTION OF PROBLEM 163, DECEMBER NUMBER.

BY J. M. ARNOLD, CROMPTON, R. I.

The published solution is correct as far as the algebraical part is concerned but when it comes to pairing the couples there is quite a mixing up of the names.

Hendricks is Anna's husband is correct, but Klaus is not Katrine's husband as stated, neither is Hans the husband of Gertrude.

The problem reduces to this: To find three square numbers, which if 63 be added to each the sum in each case will be square. These numbers are readily found by trial to be 1, 81 and 961. We can then make out the following scheme:

	Shillings	Hogs	Hogs	Shillings	
Hans	64	8	1	1	Katrine
Klaus	144	12	9	81	Gertrude
Hendricks	1024	32	31	961	Anna

We see that the third man bought 23 hogs more than the second woman. Hence Hendricks was the name of the third man and Gertrude the name of the second woman. Also the second man bought 11 more than the first woman. Hence, Klaus was the second man and Katrine the first woman. By writing in these names and giving the remaining spaces to the other two names, we shall have each man's name opposite to that of his wife. From which we see that Klaus is Gertrude's husband and not Katrine's as stated.

Also solved by *LON C. WALKER*. Professor Walker also solved No. 164.

ALGEBRA.

164. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

By Sylvester's dialytic method form the eliminant between $mx^3+py^2=0$ ....(1), and $px^2+my^3=0$....(2). Also between $mx^4+py=0$....(1), and $px^3+my^3=0$....(2).

Solution by DR. L. E. DICKSON, The University of Chicago.

The statement of this problem is unfortunate. There is no condition on the coefficients m and p in order that the either pair of equations (1) and (2) shall be simultaneous. The trivial cases in which either m or p is zero will be excluded.

(1) Solution of the simultaneous equations

$$mx^3+py^2=0,\ my^3+px^2=0.$$

Since m and p are not both zero, the determinant

$$\begin{vmatrix} x^3 & y^2 \\ y^3 & x^2 \end{vmatrix}=0, \text{ whence } x^5=y^5.$$

Let $x=\omega y$, where ω is an arbitrary fifth root of unity. Then

$$m\omega^3y^3+py^2\equiv y^2(m\omega^3y+p)=0.$$

Hence there are six sets of solutions:

$$x=0,\ y=0;\ x=\frac{p}{m}\omega^3,\ y=-\frac{p}{m}\omega^2.$$

(2) Solution of the simultaneous equations

$$mx^4+py=0,\ my^3+px^3=0.$$

In addition to the solutions $x=y=0$, there are exactly nine sets of solutions

$$x=\varepsilon p^{4/9}m^{(-4)/9},\ y=-\varepsilon^4 p^{7/9}m^{(-7)/9},$$

where ε is an arbitrary ninth root of unity.

165. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.

Solve $x^4-x=14$, by quadratics.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$x^4-x=14$ or $x^4-16=x-2$.

$$\therefore x^2-4=\frac{x-2}{x^2+4}=\frac{x}{x^2+4}-\frac{2}{x^2+4}.$$

$$\therefore x^2-\frac{x}{x^2+4}+\frac{1}{4(x^2+4)^2}=4-\frac{2}{x^2+4}+\frac{1}{4(x^2+4)^2}.$$

$$\therefore x=2 \text{ or } x=-2+\frac{1}{x^2+4}.$$

$\therefore x=2$ or $x^3+2x^2+4x+7=0$.

$\therefore x=2$ or $=-1\frac{1}{15}$ nearly, or $-\frac{1}{15}[1\pm\sqrt{(-843)}]$ nearly.

166. Proposed by MARCUS BAKER, U. S. Geological Survey, Washington, D. C.

Solve

$$ax+by=2zx....(1).$$
$$cy+dz=2xy....(2).$$
$$ez+fx=2yz....(3).$$

Solution by the PROPOSER.

From (1), (2), and (3), respectively,

$$y=\frac{(2z-a)x}{b}=\frac{dz}{2x-c}=\frac{ez+fx}{2z},$$

whence

$$x(2x-c)(2z-a)=bdz....(4),$$
$$fx(2x-c)+ez(2x-c)=2dz^2....(5).$$

From (4), $z=\frac{ax(2x-c)}{2x(2x-c)-bd}$, which substituted in (5), gives, after reduction,

$$x^4+\frac{1}{2f}(ae-2cf)x^3+\frac{1}{4f}[fc^2-a^2d-2(ace+bdf)]x^2$$

$$+\frac{1}{8f}[ac(ad+ce)+bd(2cf-ae)]x+\frac{1}{16f}bd(ace+bdf)=0.$$

Similarly,

$$y^4+\frac{1}{2b}(ac-2be)y^3+\frac{1}{4b}[fc^2-e^2b-2(ace+bdf)]y^2$$

$$+\frac{1}{8b}[ce(ae+cf)+df(2be-ac)]y+\frac{1}{16b}df(ace+bdf)=0.$$

$$z^4+\frac{1}{2d}(ce-2ad)z^3+\frac{1}{4d}[a^2d-e^2b-2(ace+bdf)]z^2$$

$$+\frac{1}{8d}[ac(ae+bc)+bf(2ad-ce)]z+\frac{1}{16d}bf(ace+bdf)=0.$$

Also solved by *LON C. WALKER*.

GEOMETRY.

REMARKS ON NO. 187, GEOMETRY, BY J. R. HITT, GOSS, MISS.

There seems to be an error in (4) of Professor Zerr's demonstration of No. 187, Geometry. The result given is not correct. For $t=\frac{1}{\sqrt{2}}b\sin C\cos C$, $t_1=\frac{1}{\sqrt{2}}b\cos C$, $t_2=\frac{1}{\sqrt{2}}b\sin C$, from which it is seen that in general t^2 cannot equal t_1t_2. It is also easily seen that if $t_1 : t=t : t_2$, then must $DI=b$, whereas DI cannot be $>\frac{1}{2}b$.

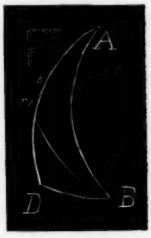
CALCULUS.

154. Proposed by B. R. DOWNER, Hopkinsville, Ky.

At the equinox, when the sun is on the celestial equator, a man starts driving on a perfectly level plain at six o'clock in the morning, and continues, going always from the sun, at the uniform rate of six miles per hour, until six o'clock in the evening. Required the path he will travel and the distance in a straight line from starting point to stopping point.

Solution by the PROPOSER.

Let the line from west to east through the starting point be taken for the axis of X, and that from south to north through the same point for the axis of Y. Let δ be the latitude of the place; then will the inclination of the celestial equator to the vertical circle passing through the east and west points be equal to δ.



Let AB be an arc of this vertical circle extending from the zenith A to the east point B, 90°. Let BC be an arc of the celestial equator, inclined at the angle δ to AB, and AC the vertical circle passing through the sun C at any particular time. If then, z represents the arc of the curve traveled by the man in any given time, BC, the sun's apparent path, expressed in terms of z, will be, for the radius 1, $\pi z/72$. Produce AC until AD is 90°, and draw BD. Then will the angles ABD and D be right angles and the arc BD will measure the angle A, the sun's angular distance at any time from the east point. Then we have from the right angled spherical triangle BCD,

$$\tan A = \sin\delta \tan\frac{\pi z}{72}.$$

But evidently the angle A is the supplement of that which the tangent at any point of the required curve makes with the axis of X.

$$\therefore \frac{dy}{dx} = -\sin\delta\tan\frac{\pi z}{72}. \quad \text{Whence } z = \frac{72}{\pi}\tan^{-1}\left(-\frac{dy/dx}{\sin\delta}\right).$$

$$\therefore dz = -\frac{72}{\pi}\frac{\sin\delta\, d^2y}{dx(\sin^2\delta + (dy^2/dx^2)}. \quad \text{But } dz = -\sqrt{(dx^2+dy^2)}.*$$

$$\therefore \sqrt{1+\frac{dy^2}{dx^2}} = \frac{72}{\pi}\frac{\sin\delta\ \dfrac{d^2y}{dx^2}}{\sin^2\delta + \dfrac{dy^2}{dx^2}}.$$

$$\text{Let } \frac{dy}{dx} = p. \quad \text{Then } \sqrt{(1+p^2)} = \frac{72}{\pi}\frac{\sin\delta(dp/dx)}{\sin^2\delta + p^2}.$$

$$\therefore dx = \frac{72}{\pi}\sin\delta\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$$

$$\text{and } pdx = dy = \frac{72}{\pi}\sin\delta\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}.$$

* The negative sign of the radical is taken because of the reverse direction in which we have measured the arc s.

Hence we have $x=\frac{72}{\pi}\sin\delta\int\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$.

Put $\sqrt{(1+p^2)}=r-p$. Then $p=\frac{r^2-1}{2r}$, $dp=\frac{r^2+1}{2r^2}dr$, $\sqrt{(1+p^2)}=\frac{r^2+1}{2r}$,

$$\sin^2\delta+p^2=\frac{r^4-2r^2\cos2\delta+1}{4r^2}.$$

Whence $\int\frac{dp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}=\int\frac{4rdr}{r^4-2r^2\cos2\delta+1}=\int\frac{4rdr}{(r^2\cos2\delta)^2+\sin^2 2\delta}$

Put $r^2-\cos2\delta=s$. Then $2rdr=ds$, and the integral becomes

$$\int\frac{2ds}{s^2+\sin^2 2\delta}=\frac{2}{\sin 2\delta}\int\frac{ds/\sin2\delta}{1+[s/(\sin2\delta)]^2}=\frac{2}{\sin 2\delta}\tan^{-1}\frac{s}{\sin 2\delta}+c.$$

$$\therefore x=\frac{72}{\pi\cos\delta}\tan^{-1}\left[\frac{p^2+p\sqrt{(1+p^2)}+\sin^2\delta}{\sin\delta\cos\delta}\right]+c.$$

But from the conditions of the problem, when $x=0$, $p=0$.

$$\therefore c=-\frac{72\delta}{\pi\cos\delta}, \text{ and } x=\frac{72}{\pi\cos\delta}\tan^{-1}\left[\frac{p^2+p\sqrt{(1+p^2)}+\sin^2\delta}{\sin\delta\cos\delta}\right]-\frac{72\delta}{\pi\cos\delta}.$$

Again we have $y=\frac{72}{\pi}\sin\delta\int\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}$.

Put $\sqrt{(1+p^2)}=r$. Then $pdp=rdr$. $\sin^2\delta+p^2=r^2-\cos^2\delta$.

$$\int\frac{pdp}{(\sin^2\delta+p^2)\sqrt{(1+p^2)}}=\int\frac{dr}{r^2-\cos^2\delta}=\frac{1}{2\cos\delta}\int\frac{dr}{r-\cos\delta}-\frac{1}{2\cos\delta}\int\frac{dr}{r+\cos\delta}$$

$$=\frac{1}{2\cos\delta}\log\left[\frac{r-\cos\delta}{r+\cos\delta}\right]+c'.$$

$$\therefore y=\frac{36\tan\delta}{\pi}\log\left[\frac{\sqrt{(1+p^2)}-\cos\delta}{\sqrt{(1+p^2)}+\cos\delta}\right]+c'.$$

But when $y=0$, $p=0$.

$$\therefore c'=-\frac{36\tan\delta}{\pi}\log\tan^2\tfrac{1}{2}\delta.$$

$$\therefore\ y=\frac{36\tan\delta}{\pi}\log\left[\frac{\sqrt{(1+p^2)}-\cos\delta}{\sqrt{(1+p^2)}+\cos\delta}\right]-\frac{36\tan\delta}{\pi}\log\tan^2\tfrac{1}{2}\delta.$$

From the two equations for x and y, we have

$$p^2+p\sqrt{(1+p^2)}=\sin\delta\cos\delta\tan\left[\frac{\pi x\cos\delta}{72}+\delta\right]-\sin^2\delta\ldots.(\text{i}).$$

$$\sqrt{(1+p^2)}=\cos\delta\frac{\cot^2\frac{1}{2}\delta+e^{\pi y/36\tan\delta}}{\cot^2\frac{1}{2}\delta-e^{\pi y/36\tan\delta}}\ldots.(\text{ii}).$$

Denote the second members of equations (i) and (ii) by X and Y. Then we have $p^2+p\sqrt{(1+p^2)}=X$, $\sqrt{(1+p^2)}=Y$. Whence, by eliminating p, we have

$$Y^2=\frac{(X+1)^2}{2X+1}\ldots.(\text{iii}).$$

which is an equation expressing the general character of the curve, which is transcendental. Making $x=0$, we may find two values of y corresponding, which are evidently the starting and stopping points in the problem proposed.

When $x=0$, $X=0$. Hence from (iii) we find $Y=\pm1$, or

$$\cos\delta\frac{\cot^2\frac{1}{2}\delta+e^{\pi y/36\tan\delta}}{\cot^2\frac{1}{2}d-e^{\pi y/36\tan\delta}}=\pm1;$$

whence $e^{\pi y/36\tan\delta}=1$ or $\cot^4\frac{1}{2}\delta$.

$\therefore\ y=0$, or $\frac{144\tan\delta}{\pi}\log\cot\frac{1}{2}\delta$, and the distance from starting point to stopping point is the latter value of y.

For the latitude 36° 20′ we have for the distance from starting point to stopping point 37.56 miles.

MECHANICS.

144. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Pressure is applied perpendicularly to the plane surface yz, bounding an otherwise infinite isotropic solid. Find the resultant displacements, if the pressure varies as $\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)$.

Solution by the PROPOSER.

Measuring the axis of x perpendicularly to the plane face into the interior of the solid, the longitudinal stresses P, Q and the shearing stress T are functions of x and y only, and the general equations of internal equilibrium reduce to (Thompson and Tait, *Natural Philosophy*, Art. 697) the following:

$$\left.\begin{aligned} dP/dx+dT/dy=0 \\ dT/dx+dQ/dy=0 \end{aligned}\right\}\ldots.(1)$$

Let α, β be the displacements of the point originally at (x, y), k=the elasticity of volume, and n=the elasticity of figure of the substance, and let $m=k+\frac{1}{3}n$, then

$$\left.\begin{aligned} P&=(m+n)d\alpha/dx+(m-n)d\beta/dy \\ Q&=(m-n)d\alpha/dx+(m+n)d\beta/dy \\ T&=n(d\beta/dx+d\alpha/dy) \end{aligned}\right\}\ldots.(2).$$

Let $\delta=d\alpha/dx+d\beta/dy$=the superficial dilation, and $\nabla^2=d^2/dx^2+d^2/dy^2$, then equations (1) become

$$\left.\begin{aligned} md\delta/dx+n\nabla^2\alpha=0 \\ md\delta/dy+n\nabla^2\beta=0 \end{aligned}\right\}\ldots.(3).$$

Assume $P=\varphi(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]$

$$Q=\psi(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]$$

$$T=\theta(x)\left[\cos\left(\frac{2\pi y}{a}\right)+\cosh\left(\frac{2\pi y}{a}\right)\right].$$

From equations (1),

$$\varphi'(x)\left[\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)\right]-\frac{2\pi}{a}\theta(x)\left[\sin\left(\frac{2\pi y}{a}\right)-\sinh\left(\frac{2\pi y}{a}\right)\right]=0.$$

$$\theta'(x)+\frac{2\pi}{a}\psi(x)=0.$$

$$\therefore\ \theta(x)=\frac{a}{2\pi}\varphi'(x)\left[\frac{\sin\left(\frac{2\pi y}{a}\right)+\sinh\left(\frac{2\pi y}{a}\right)}{\sin\left(\frac{2\pi y}{a}\right)-\sinh\left(\frac{2\pi y}{a}\right)}\right]=\frac{a}{2\pi}B\varphi'(x),\text{ suppose.}$$

$$\therefore\ \theta'(x)=\frac{a}{2\pi}\varphi''(x)B.$$

$$\therefore\ \psi(x)[\sin(2\pi y/a)-\sinh(2\pi y/a)]+(a^2/4\pi^2)\varphi''(x)[\sin(2\pi y/a)$$
$$+\sinh(2\pi y/a)]=0,\text{ or }\psi(x)+B(a^2/4\pi^2)\varphi''(x)=0.$$

From (2), $du/dx=(P+Q)/4m+(P-Q)/4n$,
$d\beta/dy=(P+Q)/4m-(P-Q)/4n$,

$$\delta=\frac{1}{2m}(P+Q)=\frac{1}{2m}[\varphi(x)+\psi(x)][\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

If $P=pF(y)$ represent the distribution of traction and pressure on the plane face, then $\delta=P/m=(p/m)F(y)$, when $x=0$, and from equations (3). $\nabla^2\delta=0$; δ must also vanish when $x=\infty$.

$$\therefore\ \nabla^2\delta=(1/2m)\{[\varphi''(x)+\psi''(x)][\sin(2\pi y/a)+\sinh(2\pi y/a)]$$
$$-(4\pi^2/a^2)[\varphi(x)+\psi(x)][\sin(2\pi y/a)-\sinh(2\pi y/a)]\}=0.$$

$$\therefore\ B[\varphi''(x)+\psi''(x)]-(4\pi^2/a^2)[\varphi(x)+\psi(x)]=0.$$

$$\therefore B[\varphi''(x)-B(a^2/4\pi^2)\varphi''''(x)]-(4\pi^2/a^2)[\varphi(x)-B(a^2/4\pi^2)\varphi''(x)]=0.$$

$$\therefore\ 2B\varphi''(x)-(4\pi^2/a^2)\varphi(x)-B^2(a^2/4\pi^2)\varphi''''(x)=0.$$

Let $\varphi(x)+\psi(x)=\varphi(x)-B(a^2/4\pi^2)\varphi''(x)=f(x)$.

$$\therefore\ Bf''(x)-(4\pi^2/a^2)f(x)=0.\quad\therefore f(x)=Ce^{-(2\pi x/a\sqrt{B})}.$$

Let $2\pi/a\sqrt{B}=b$. $\therefore\ \varphi''(x)-b^2\varphi(x)+b^2Ce^{-bx}=0$.

The solution of this equation is $\varphi(x)=De^{-bx}-\frac{1}{2}bCxe^{-bx}$.

$$\therefore\ \theta(x)=-\frac{abB}{2\pi}De^{-bx}-\frac{abB}{4\pi}Ce^{-bx}+\frac{ab^2B}{4\pi}Cxe^{-bx}.$$

$$\psi(x)=-De^{-bx}-Ce^{-bx}+\tfrac{1}{2}bCxe^{-bx}.$$

Now when $x=0$, $P=Q=p[\sin(2\pi y/a)+\sinh(2\pi y/a)]$. $T=0$; therefore $\varphi(0)=\psi(0)=p$, $\theta(0)=0$; $\therefore\ D=p$ and $C+2D=0$ or $C=-2p$.

$$\therefore\ P=pe^{-bx}(1+bx)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$Q=pe^{-bx}(1-bx)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$T=-\frac{2\pi dx}{a}e^{-bx}[\cos(2\pi y/a)+\cosh(2\pi y/a)].$$
$$\delta=(p/m)e^{-bx}[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

$$du/dx=\tfrac{1}{2}pe^{-bx}(1/m+bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$d\beta/dy=\tfrac{1}{2}pe^{-bx}(1/m-bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$\alpha=-(p/2b)e^{-bx}[1/m+1/n(1+bx)][\sin(2\pi y/a)+\sinh(2\pi y/a)].$$
$$\beta=\tfrac{1}{2}p\int e^{-bx}(1/m-bx/n)[\sin(2\pi y/a)+\sinh(2\pi y/a)].$$

In this last, $b=\dfrac{2\pi}{a}\sqrt{\dfrac{\sin(2\pi y/a)-\sinh(2\pi y/a)}{\sin(2\pi y/a)+\sinh(2\pi y/a)}}$.

AVERAGE AND PROBABILITY.

125. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

A circle with unknown radius is described with its center at the extremity of the major axis of a given ellipse. Required, the average area common to the circle and the ellipse.

Solution by the PROPOSER.

Take for equations of the ellipse and variable circle, respectively,

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1....(1), \text{ and } x^2+y^2=r^2....(2).$$

From (1), (2), and the properties of the ellipse with the usual notation, we find, after reducing,

$$r=2a\sin\tfrac{1}{2}\theta\sqrt{[1-\varepsilon^2\cos^2\tfrac{1}{2}\theta]}....(3),$$

and $\alpha=2\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right)$, the angle formed by the radii drawn to the points of intersection of the ellipse and circle. Then the area common to both is

$$S=\int \alpha r dr=4a^2\sin^2\tfrac{1}{2}\theta(1-\varepsilon^2\cos^2\tfrac{1}{2}\theta)\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right)+ab\theta-ab\sin\theta....(4).$$

Hence the required average area is

$$A=\frac{1}{2a}\int_0^{2a} S dr=\frac{1}{2a}\left[rS-\int r dS\right]_0^{2a}....(5).$$

Now substituting in (5) the value of r from (3), the values of S and dS from (4), integrating, and observing that when $r=0$, $\theta=0$; and when $r=2a$, $\theta=\pi$, we finally have

$$A=\frac{1}{2a}\Big[\tfrac{8}{3}a^3\sin^3\tfrac{1}{2}\theta[1-\varepsilon^2\cos^2\tfrac{1}{2}\theta]^{\frac{3}{2}}\tan^{-1}\left(\frac{b\cot\frac{1}{2}\theta}{a}\right)+2a^2b\theta\sin\tfrac{1}{2}\theta[1-\epsilon\cos^2\tfrac{1}{2}\theta]^{\frac{1}{2}}$$

$$-2a^2b\sin\theta\sin\tfrac{1}{2}\theta[1-\epsilon^2\cos^2\tfrac{1}{2}\theta]^{\frac{1}{2}}$$

$$-\frac{2a^2b}{3\varepsilon^2}[4\varepsilon^2-1]\cos^{-1}[\varepsilon\cos\tfrac{1}{2}\theta]+\frac{2a^2b}{3\varepsilon^2}[4\varepsilon^2+1]\cos\tfrac{1}{2}\theta[1-\epsilon^2\cos^2\tfrac{1}{2}\theta]^{\frac{1}{2}}$$

$$-\frac{4a^2b}{3}\cos^3\tfrac{1}{2}\theta[1-\varepsilon^2\cos^2\tfrac{1}{2}\theta]^{\frac{1}{2}}\Big]_0^{\pi}=ab\left[\pi-\left(\frac{4\varepsilon^2-1}{3\varepsilon^3}\right)\sin^{-1}\varepsilon-\frac{b}{a}\left(\frac{2\varepsilon^2+1}{3\varepsilon^2}\right)\right].$$

Also solved by G. B. M. ZERR. If space permits his solution will appear in the next issue.

DIOPHANTINE ANALYSIS.

101. Proposed by HARRY S. VANDIVER, Bala, Pa.

Prove that it is impossible to find integral values for x, y, and z such that the relation $x^2y+xz^2=y^2z$ is satisfied.

II. Solution by W. F. KING, Ottawa, Canada.

We may assume that x, y, and z have no factor common to them all. For if they have a common factor n, each term of the equation $x^2y+xz^2=y^2z$ may be divided by n^3, and there will then be left an equation of the same form, in which x, y, and z have no common factor.

Let the greatest common measure of x and y be a; of y and z, b; and of z and x, c. Then we shall have

$$x=cax_1,$$
$$y=aby_1,$$
$$z=bcz_1.$$

Now observe that since b is the greatest common measure of y and z, ay_1 is prime to cz_1; similarly is bz_1 to ax_1, and cx_1 to by_1.

Hence a, b, and c are all prime to one another; so are x_1, y_1, and z_1 to one another. Also a is prime to z_1, b to x_1, and c to y_1. Substituting in the equation the above values of x, y, and z, and dividing by abc, we have

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1.$$

Divide by x_1; then $a^2c\,x_1y_1+c^2b\,z_1^2=\dfrac{b^2a\,y_1^2z_1}{x_1}$.

The left hand side is integral, therefore $b^2a\,y_1^2z_1$ is divisible by x_1. But as above shown, x_1 is prime to b, y_1 and z_1. Hence a/x_1 must be an integer.

Again, divide the above equation by a. Then

$$acx_1^2y_1+\frac{c^2bz_1^2x_1}{a}=b^2y_1^2z_1.$$

Hence, as before, $\dfrac{c^2b\,z_1^2x_1}{a}$ is an integer. But a is prime to b, c, and z_1. Therefore, x_1/a is an integer. Now since a/x_1 and x_1/a are both integers, $x_1=a$. Similarly, $y_1=b$, $z_1=c$. Therefore the equation

$$a^2c\,x_1^2y_1+c^2b\,z_1^2x_1=b^2a\,y_1^2z_1$$

becomes $a^4bc+abc^4=ab^4c$, or dividing by abc, $a^3+c^3=b^3$, a known impossible form.

102. Proposed by F. L. SAWYER. Mitchell, Ontario, Canada.

Prove that the factors of the sum of the squares of two numbers prime to each other are themselves the sum of two squares.

Solution by DR. L. E. DICKSON, The University of Chicago.

If a and b are relatively prime, every prime divisor of a^2+b^2 is of the form $4n+1$; inversely, every prime of the form $4n+1$ can be expressed as the sum of the squares of the two relatively prime integers. These are well known theorems in the Theory of Numbers (compare Weber's *Algebra*, 1st edition, I, p. 585). Let a^2+b^2 have the prime factors $p_1, p_2, \ldots, p_s$, which need not be distinct. In view of the theorems quoted, $p_i=x_i^2+y_i^2$, where x_i and y_i are integers. But

$$(x_i^2+y_i^2)(x_j^2+y_j^2)=(x_ix_j+y_iy_j)^2+(x_iy_j-x_jy_i)^2.$$

It fellows that any prime or composite divisor of a^2+b^2 is the sum of two squares.

Also solved by *LON C. WALKER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

165. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A borrows \$2000 and agrees to pay back principal and interest in 100 equal monthly payments. Find the monthly payment. What would he have to pay yearly on the same conditions in order to discharge the debt in 100 months?

ALGEBRA.

173. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve $\sqrt{(a+x+y)}=z....(1)$, $\sqrt{(b+y+z)}=x....(2)$, $\sqrt{(c+z+x)}=y....(3)$.

GEOMETRY.

195. Proposed by F. L. SAWYER, Mitchell, Ontario, Canada.

The diagonals of a four-sided figure are h and k, and the area is A; show that the area of the circumscribing square is

$$\frac{h^2k^2-4A^2}{h^2+k^2-A}.$$

196. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a quadrilateral circumscribe a circle, the two diagonals and the two lines joining the points where the opposite sides of the quadrilateral touch the circle will all four meet in a point.

CALCULUS.

161. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering in the Agricultural and Mechanical College of Texas, College Station, Texas.

A cylindrical oil tank of length l and radius r is capped by curved ends and rests with the axis horizontal. The total axial length of tank is $l+2h$. If the oil stands at depth d in the tank (d less than $2r$) find its volume (a) when the ends are portions of the surface of a sphere, (b) when the ends are portions of the surface of an ellipsoid.

A special case of the above problem was recently received from Houston, Texas, for solution. It is passed on to the readers of the MONTHLY.

162. Proposed by J. E. SANDERS, Hackney, O.

Solve the differential equations

$$(a)\ x\frac{dy}{dx}-y=x\sqrt{(x^2+y^2)},\quad (b)\ \cos x\frac{dy}{dx}+y=1-\sin x.$$

MECHANICS.

151. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

An elastic ball is projected along a horizontal tube, striking first the bottom, then the top, then the bottom, and so on. Find the number of times the ball will strike the top.

152. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

A ball of lead an inch in diameter is cast into a ball of rubber two inches in diameter so as to be internally tangent. What is the nature of the path made by this *lead-rubber* ball in rolling down an inclined plane?

DIOPHANTINE ANALYSIS.

112. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

There is a series of rational triangles whose sides have a common difference of unity. Calling the one whose sides are 3, 4, 5, the first triangle, find the sides of the nth triangle.

AVERAGE AND PROBABILITY.

136. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

By direct computation find the average distance between two points in the surface of a given rectangle, but on opposite sides of a diagonal.

137. Proposed by G. H. HARVILL, Malakoff, Texas.

A, B, C, and D, in playing whist, agree that the person who first cuts an ace shall have a stake of \$313. What is the value of each person's expectation before the play begins, each taking his turn at cutting in the order named as the game progresses?

NOTES.

Professor Luigi Cremona, of Rome, has been elected a foreign member of the American Academy of Arts and Sciences, of Boston. F.

Any of our readers knowing of a vacancy in mathematics, or of a prospective vacancy, will confer a favor on us by informing us of the same. We are in a position to recommend a number of most excellent candidates for such vacancies. F.

On December 30th, 1902, editor, Dr. L. E. Dickson, was married to Miss Susan M. Davis, of Waco, Texas. After the wedding, the fortunate couple started on an extended trip through Old Mexico, remaining several weeks at the City of Mexico. Congratulations are now in order. F.

BOOKS AND PERIODICALS.

The Elements of Plane and Spherical Trigonometry. By T. U. Taylor, C. E., University of Texas, and C. Puryear, M. A., C. E., Agricultural and Mechanical College of Texas. Large 8vo. Cloth. 160 pages of text, 67 pages of tables. Price, $1.25. 1902. Boston: Ginn & Co.

The authors have succeeded in realizing their aim to present the essential parts of trigonometry in a form adapted to practical applications. The chief novelty lies in the excellent sets of new exercises and problems of a practical character. Some disappointment is felt over the omission of the projective proofs of the addition formulae. On page 46 functions are indicated with the angle omitted. Napier's rules are not merely an aid to the memory, but of added value in that they enable us to pick out without device that one of the ten formulae needed in a given case. While noting the occurance of the vowels *a* and *o* in Napier's Rules, no mention is made of the vowel *i* in *sine* and *middle*. The book appears to possess pedagogical excellence, and the typography is attractive. D.

Solid Geometry. Revised Edition. By G. A. Wentworth. 8vo. xvi+218 pages. Price $0.75. 1902. Boston: Ginn & Co.

The book is a reprint of pages 251-459 of Wentworth's revised Plane and Solid Geometry (1899), with modified table of formulas and index, and ten introductory pages giving the necessary references to plane geometry. D.

An Elementary Treatise on the Mechanics of Machinery with special reference to the Mechanics of the Steam Engine. By Joseph N. Le Conte, Instructor in Mechanical Engineering, University of California, Associate member of the American Institute of Electrical Engineers, etc. 8vo. Cloth, x+311 pages. Price, $2.25. New York: The Macmillan Co.

The book is divided into three parts. The first two embody the more important principles of what may be called the Kinematics of Machinery, and the third part treats of the Mechanics of the Steam Engine, kinematically and dynamically. The work presents in a most excellent manner the applications of the principles of mechanics to certain prob-

lems connected with machinery. All the problems are clearly worked out and illustrated with diagrams neatly constructed. This book will prove of great value both to the theoretical mechanic and the practical machinist. F.

Elementary Applied Mechanics. By T. Alexander, C. E., M. Inst. C. E. I., M. A. I. (Hon. Causa), 4th Order of Meiji, Japan, Professor of Engineering, Trinity College, Dublin, and A. W. Thomson, D. Sc., Professor of Engineering, College of Science, Poona. With numerous diagrams, and a series of graduated examples carefully worked out. 8vo. Cloth, xx+575 pages. Price, $5.25. New York: The Macmillan Co.

In this work the following subjects are treated very fully: Internal stress and strain; transverse stress; bending moments, and shearing forces for fixed loads, for combined fixed loads, and for moving loads; resistance, in general, to bending and shearing at the various cross-sections of framed girders and solid beams; stress at an internal point of a beam; curvature, slope, and deflection; and a number of other allied subjects. Each subject treated is fully illustrated with the solutions of a number of problems bearing on it. The work forms an elementary consecutive course on the subject of internal stress and strain, based on the late Professor Rankine's treatment of the subject in his *Applied Mechanics* and *Civil Engineering*. It is the best elementary treatise that has yet appeared. F.

Manual of Advanced Optics. By C. Riborg Mann, Assistant Professor of Physics in the University of Chicago. 8vo. Cloth, 196 pages. Price, $2.00. Chicago: Scott, Forseman & Co.

The object of this work is to meet the needs of the more advanced students of Optics. While there are many excellent works treating the theory of optics very exhaustively, yet none treat of some of the recent and important discoveries. No work thus far, to my knowledge, treats of the practical applications of that marvelous little instrument, the Interferometer, invented by Dr. Michelson. In the work before us is found not only the treatment of the adjustment and use of the interferometer, but also the use of defracting and concave gratings, the prism, spectrometer, etc. Dr. Mann has the happy faculty of clearing up the difficulties of the subject, and this book will be greatly appreciated by all students interested in the subject of Optics. F.

The American Journal of Mathematics. Edited by Frank Morley and others. Published quarterly, under the auspices of the Johns Hopkins University. Price, $5.00 per year in advance.

No. 1 of Vol. XXV contains the following articles: The Parametric Representation of the Tetrahedroid Surface, by D. M. Lehmer; On Ternary Monomial Substitution-Groups of Finite Order with Determinant + or −1, by E. B. Skinner; On the Forms of Unicusal Sextic Scrolls, and On the Forms of Sextic Scrolls of Genus One, by Virgil Snyder; Note on Symmetric Functions, by E. D. Roe, Jr. F.

The Annals of Mathematics. Edited by Ormond Stone, W. E. Byerly, and others. Published Quarterly under the auspices of Harvard University. Price, $2.00 per year in advance.

No. 2 of Vol. 4, second series, contains the following articles: The Logarithm as a Direct Function, by J. W. Bradshaw, with an Introduction by W. F. Osgood; On Positive Quadratic Forms, by Paul Saurel; Multiple Points on Lissajous's Curves in Two and Three Dimensions, by Edward A. Hook; A Special Quadri-Quadric Transformation of Real Points in a Plane, by C. C. Engberg. F.

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AN ELEMENTARY EXAMPLE OF MODULAR SYSTEMS.

By PROF. G. A. MILLER.

The following exposition is based upon Kronecker's *Lectures on Number Theory*, pp. 144-150.* While the example is very elementary and does not lead to any new results, yet it may serve to give some idea of the general modular systems, and also to exhibit a fundamental theorem of arithmetic in an attractive manner.

The number† a is said to be divisible by m whenever it is possible to find a number c such that $a=cm$. Similarly we may say that a is divisible by the system of numbers $m_1, m_2, \ldots, m_\alpha$ provided it is possible to find α numbers $c_1, c_2, \ldots, c_\alpha$ such that

$$a=c_1 m_1+c_2 m_2+\ldots+c_\alpha m_\alpha .$$

From this standpoint the system of numbers $m_1, m_2, \ldots, m_\alpha$ is called a *modular system* and it is denoted by $(m_1, m_2, \ldots, m_\alpha)$. The numbers $m_1, m_2, \ldots, m_\alpha$ are called the *elements* of the system. For instance, 3 is divisible by the modular system (7, 16, 25) because $3=3.7+2.16-2.25$. In particular, zero is divisible by every modular system since

$$0=0.m_1+0.m_2+\ldots+0.m_\alpha .$$

Vorlesungen ueber Zahlentheorie von Leopold Kronecker, Erster Band, Leipzig, 1901, B. G. Teubner.

†Only integers, including 0, are meant by the term *number* as used in this note.

One modular system $(m_1, m_2, \ldots, m_\alpha)$ is said to be divisible by another $(d_1, d_2, \ldots, d_\beta)$ whenever each element of the first system is divisible by the second system; *i. e.*, when the following α equations are satisfied:

$$\begin{aligned} m_1 &= c_{11}d_1 + c_{12}d_2 + \ldots + c_{1\beta}d_\beta , \\ m_2 &= c_{21}d_1 + c_{22}d_2 + \ldots + c_{2\beta}d_\beta , \\ &\cdots\cdots\cdots \\ m_\alpha &= c_{\alpha 1}d_1 + c_{\alpha 2}d_2 + \ldots + c_{\alpha\beta}d_\beta . \end{aligned}$$

For instance, the system (20, 35) is divisible by the system (15, 25, 10) since

$$\begin{aligned} 20 &= 0.15 + 0.25 + 2.10, \\ 35 &= 1.15 + 0.25 + 2.10. \end{aligned}$$

From the following equations it follows that the system (15, 25, 10) is also divisible by the system (20, 35):

$$\begin{aligned} 15 &= 1.35 - 1.20, \\ 25 &= 3.35 - 4.20, \\ 10 &= 2.35 - 3.20. \end{aligned}$$

Two modular systems which have the property that each of them is divisible by the other are called *equivalent*. The above examples prove that the two systems (20, 35) and (15, 25, 10) are equivalent. This relation may be indicated by the symbol $(20, 35) \backsim (15, 25, 10)$. It may be observed that we are here dealing with an interesting *extension of ordinary division*. If each of two equivalent modular systems is composed of only one element these two elements must be equal as in ordinary division.

If a number is divisible by $(m_1, m_2, \ldots, m_\alpha)$ it is evidently also divisible by $(d_1, d_2, \ldots, d_\beta)$, since the latter is a divisor of the former. In particular, if a number is divisible by a given modular system it is also divisible by every equivalent system. If each one of two numbers is divisible by $(m_1, m_2, \ldots, m_\alpha)$ it is evident that their sum and their difference are divisible by the same system; *i. e.*, the multiples of a given modular system reproduce themselves when they are combined with respect to the two operations, addition and subtraction. Hence they must also reproduce themselves with respect to multiplication.

Since the multiples of two equivalent systems are identical, it is of interest to determine the simplest system which is equivalent to a given system $(m_1, m_2, \ldots, m_\alpha)$. It is not difficult to see directly that this consists of a single element, viz. the greatest common divisor of the elements $m_1, m_2, \ldots, m_\alpha$. That is, *each modular system of numbers is equivalent to some modular system composed of one element*. If the elements of the modular system are functions of one or more